NonAbelian states with negative flux : a new series of quantum Hall states

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By applying the idea of parafermionic clustering to composite bosons with positive as well as *negative* flux, a new series of trial wavefunctions to describe fractional quantum Hall states is proposed. These states compete at filling factors $\nu = k/(3k \pm 2)$ with other ground states like stripes or composite fermion states. These series contain all the states recently discovered by Pan et al. [Phys. Rev. Lett. **90**, 016801 (2003)] including the even denominator cases. Exact diagonalization studies on the sphere and torus point to their relevance for $\nu = 3/7, 3/11$, and $3/8$.

Two-dimensional electron gases in a quantizing magnetic field display a wealth of incompressible liquid phases at low temperature. These liquids may be classified by sequences of special values of Landau level (LL) filling factors ν. The most prominent sequences are observed for $\nu = p/(2mp \pm 1)$, p and m integers. In the composite-fermion picture [1, 2, 3, 4, 5], this is interpreted as an integer Hall effect for composites made of an electron bound to $2m$ flux quanta, the composite fermions ^{2m}CFs.

Recent experiments [6] have uncovered states displaying the fractional quantum Hall effect (FQHE) at filling factors $\nu = 4/11, 5/13, 4/13, 6/17$, and $5/17$ that do not belong to the primary FQHE sequences. In addition, there is also evidence for two even-denominator fractions $\nu = 3/10$, and $3/8$. This is very unusual since the only previously known example of an even-denominator fraction is the elusive $\nu = 5/2$ state. The state $3/8$ has also been observed [7] in the N=1 LL at total filling factor $\nu = 2 + 3/8$. The new odd-denominator fractions can be explained by hierarchical reasoning in the spirit of the original Halperin-Haldane hierarchy. For example, at $\nu = 4/11$, the ^{2}CFs have an effective filling factor $\nu_{CF} = 1 + 1/3$. If the interactions between the ^{2}CFs have a repulsive short-range core then it is plausible that they will themselves form a standard Laughlin liquid at filling factor $1/3$ within the second CF Landau level. This possibility pointed out in the work of Pan et al. [6] has been explored theoretically [8, 9, 10]. It should be pointed out that this construction of "second generation" of composite fermions is part of the standard lore of the hierarchical view of the FQHE states since the CF construction and the older Halperin-Haldane hierarchy can be related by a change of basis in the lattice of quantum numbers [11]. Since the even-denominator fractions requires clustering they do not fit naturally in this picture.

In this Letter, I propose a construction based on the idea of composite bosons that carry now an odd number of flux quanta. These fluxes may be positive or negative. I then exploit the possibility of clustering of bosons in the lowest LL (LLL). Indeed it has been suggested [12] that incompressible liquids of Bose particle may form at fill-

ings $\nu = k/2$ with integer k. I write down spin-polarized FQHE wavefunctions on the disk and spherical geometry. By construction they reside entirely in the LLL and have filling factor $\nu = k/(3k \pm 2)$. All these states are non-Abelian FQHE states with unconventional excitations. While the positive flux series already appeared in the work of Read and Rezayi [13, 14], the negative flux series is new. These series produce candidate wavefunctions for all the states observed by Pan et al. beyond the main CF sequences, thus unifying even and odd denominator fractions. For the fraction $3/7$, the negative flux candidate wavefunction has an excellent overlap with the Coulomb ground state obtained by exact diagonalization on the sphere for N=6 electrons. There is also a two-fold possible topological degeneracy on the torus geometry obersevd by tweaking the Coulomb interaction. The possible non-Abelian $3/11$ state is also observed on the sphere. Finally at $\nu = 3/8$ there is a seven-fold quasi degeneracy of the ground state on the torus also compatible with a non-Abelian state.

The first observation is that some of the new fractions of ref.([6]) are of the form $p/(3p \pm 1)$. This would be natural for the FQHE of *bosons* where one expects the formation of composite fermions with an odd number of flux tubes, i.e. ^{1}CF and ^{3}CF. The ^{1}CF lead to a series of Bose fractions at $\nu = p/(p + 1)$ which has nothing to do with the present problem. But if the ^{3}CFs fill an integer number of pseudo-Landau levels then this leads to magic fillings $p/(3p \pm 1)$. Indeed there is evidence from theoretical studies of bosons in the LLL with dipolar interactions [15] that such ^{3}CF do appear. This suggests that composite bosons may form in the electronic system, three flux tubes bound to one electron, ^{3}CBs, the attachment may be with statistical flux along or against the applied magnetic field. If ν stands for the electron filling factor and ν^* the ^{3}CB filling factor, they are related by $1/\nu = 3 + 1/\nu^*$. The relationship between the wanted electronic trial wavefunction and the CB wavefunction is :

$$\Psi_\nu^{Fermi}(\{z_i\}) = \mathcal{P}_{LLL} \prod_{i<j} (z_i - z_j)^3 \, \Phi_{\nu^*}^{Bose}(\{z_i, z_i^*\}), \quad (1)$$

where $z_i = x_i + iy_i$ refer to the coordinates of the electrons in the unbounded disk geometry and the symmetric gauge, $\mathcal{P}_{LLL}$ is the LLL projection operator. The Laughlin-Jastrow factor $\prod_{i<j}(z_i - z_i)^3$ transforms bosons into fermions and adequately takes into account the Coulomb repulsion. The next step is to find candidates for the trial state $\Phi_{\nu^*}^{Bose}$. It has been suggested [12] that bosons in the LLL may form incompressible states for $\nu^* = k/2$. There is evidence that they are described by the Read-Rezayi parafermionic states [13, 14] with clustering of k particles :

$$\Phi_{\nu^*=k/2}^{RR} = \mathcal{S} \left[\prod_{i_1<j_1} (z_{i_1} - z_{j_1})^2 \dots \prod_{i_k<j_k} (z_{i_k} - z_{j_k})^2 \right].$$ (2)

In this equation, the $\mathcal{S}$ symbol means symmetrization of the product of Laughlin-Jastrow factors over all partition of N particles in subsets of N/k particles (N being divisible by k). The ubiquitous exponential factor appearing in all LLL states has been omitted for clarity. While the relevance of such states to bosons with contact interactions is not clear, it has been shown that longer-range interactions like dipolar interaction may help stabilize these states [15]. Since the CBs are composite objects it is likely that their mutual interaction has also some long-range character. It is thus natural to try the ansatz $\Phi_{\nu^*}^{Bose} = \Phi_{\nu^*=k/2}^{RR}$ in Eq.(1). This leads to a series of states with electron filling factor $\nu = k/(3k+2)$ which is in fact the $M = 3$ case of the generalized (k, M) states constructed by Read and Rezayi. In this construction, the flux attached to the boson is positive. It is also possible to construct wavefunctions with negative flux [16] attached to the CBs. Now the Bose function depens only upon the antiholomorphic coordinates :

$$\Phi_{\nu^*}^{Bose}(\{z_i^*\}) = (\Phi_{\nu^*=k/2}^{RR}(\{z_i\}))^*$$ (3)

The projection onto the LLL in Eq.(1) means that the electronic wavefunction can be written as :

$$\Psi_\nu^{Fermi}(\{z_i\}) = \Phi_{\nu^*=k/2}^{RR}(\{\frac{\partial}{\partial z_i}\}) \prod_{i<j}(z_i - z_j)^3.$$ (4)

The filling factor of this new series of states is now $\nu = k/(3k-2)$. These states can be written in the spherical geometry with the help of the spinor components $u_i = \cos(\theta_i/2)e^{i\varphi_i/2}$, $v_i = \sin(\theta_i/2)e^{-i\varphi_i/2}$ ($\{\theta_i, \phi_i\}$ being standard polar coordinates) by making the following substitutions :

$$z_i - z_j \to u_i v_j - u_j v_i, \quad \partial_{z_i} - \partial_{z_j} \to \partial_{u_i}\partial_{v_j} - \partial_{v_i}\partial_{u_j}.$$ (5)

This construction leads to wavefunctions that have zero total angular momentum $L = 0$ as expected for liquid states. On the sphere the two series of states have a definite relation between the number of flux quanta through

the surface and the number of electrons. The positive flux series has $N_\phi = N/\nu - 5$ while the negative flux series has $N_\phi = N/\nu - 1$. Even when these states have the same filling factor as standard hierarchy/composite fermion states, the shift (the constant term in the $N_\phi - N$ relation) is in general different. The positive flux series starts with the Laughlin state for $\nu = 1/5$ at $k = 1$, the $k = 2$ state is the known Pfaffian state [17, 18] at $\nu = 1/4$, at $k = 3$ there a state with $\nu = 3/11$ which compete with the ^{4}CF state with negative flux, at $k = 4$ the competition is with the similar $\nu = 2/7$ ^{4}CF state. This series also contains $5/17$ at $k = 5$, $3/10$ at $k = 6$, and $4/13$ at $k = 8$. The negative flux series starts with the filled Landau level at $k = 1$ and contains notably $5/13$ ($k = 5$), $3/8$ ($k = 6$), $4/11$ ($k = 8$), $6/17$ ($k = 12$). It is not likely that these states will compete favorably with the main sequence CF states in view of their remarkable stability. However the situation is open concerning the exotic even denominator and the unconventional odd-denominator states. Also the CF states may be destabilized by tuning the interaction potential. A two-body interaction in a given LL may always be parameterized by the pseudopotentials V_m, $m = 1, 3, \dots$, where V_m is the interaction energy for a single pair of electrons with relative angular momentum m (all energies will be expressed in units of $e^2/\epsilon l_0$ and $l_0 = \sqrt{\hbar c/eB}$). It is known for example that there is a window of stability for a non-Abelian $\nu = 2/5$ state in the N=1 LL [19] which is obtained by slightly decreasing the V_1 component with respect to its Coulomb value.

I now show that a similar phenomenon happens at $\nu = 3/7$ in the LLL. The conventional CF state at this filling factor is a member of the principal sequence of states. It is realized for $N = 9$ electrons at $N_\phi = 16$ in the spherical geometry. There is a singlet ground state and a well-defined branch of neutral excitations for $L = 2, 3, 4, 5$: see Fig.(1a). The negative-flux state Eq.(4) requires $N_\phi = 20$ for the same number of particles. At this flux for pure Coulomb interaction there is simply a set of nearly degenerate states without evidence for an incompressible state : see Fig.(1b). If the pseudopotential V_1 is decreased from its Coulomb LLL value, the CF state is quickly destroyed (Fig.(1c)) but there is appearance of a possibly incompressible state precisely at the special shift predicted above : Fig.(1d). There is a $L = 0$ ground state and a branch of excited states for $L = 2, 3, 4$. To check if this state is really the new negative flux state proposed above, the overlap between the candidate wavefunction for $k = 3$ in Eq.(4) and the numerically obtained ground state is displayed in Fig.(2) for N=6 electrons at $N_\phi = 15$. Even for the pure Coulomb interaction the squared overlap is 0.9641 and it rises up to 0.99054 for $V_1 = 0.885 V_1^{Coulomb}$. To investigate the competition between these states it is convenient to use the torus geometry for which these is no shift. Translation symmetries of the many-body problem may be used
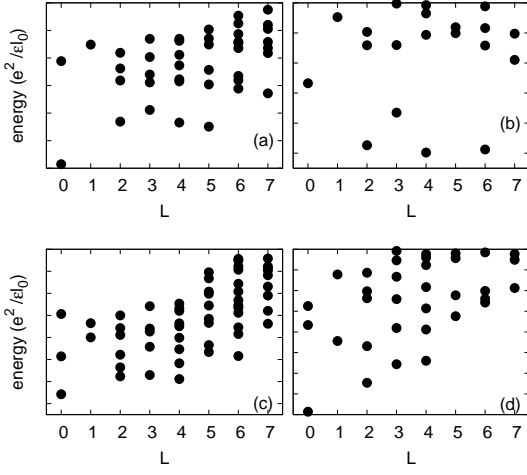
Figure 1: Low-lying spectrum of 9 electrons in the LLL as a function of the angular momentum on a sphere. Top panel : Coulomb interaction. (a) At $N_\phi = 16$, the flux needed for the 3/7 CF state there is a singlet ground state and a branch of collective excitations. (b) At the flux needed for the candidate state there is no evidence of a FQHE state. Bottom panel weakened potential with $V_1 = 0.7 V_1^{Coulomb}$: (c) The 3/7 CF state is now compressible. (d) There is a possible new FQHE state with the shift required by Eq(4).
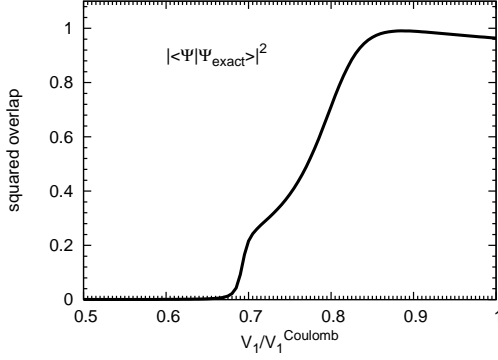

Figure 2: The squared overlap for N=6 electrons at $N_\phi = 15$ between the candidate wavefunction at $\nu = 3/7$ and the exact ground state computed by varying the pseudopotential V_1 with respect to the Coulomb value.

Figure 3: Low-lying spectrum for N=12 electrons at $N_\phi = 28$ for weakened pseudopotential $\delta V_1 = -0.125$ in a square unit cell vs pseudomomentum $K = |\mathbf{K}|$.

Figure 4: The spectrum on a sphere for $N = 6$ electrons and $N_\phi = 17$ where an unconventional state with $\nu = 3/11$ may appear. (a) with pure Coulomb interaction, there is a set low-lying states best viewed as two quasiholes on top of a $\nu = 1/3$ liquid. (b) with $V_1 = 0.65 V_1^{Coulomb}$ the $L = 0$ ground state is separated from other higher-lying state and there is a branch of states at $L = 2, 3, 4$

to construct [20] a conserved momentum $\mathbf{K} = (K_x, K_y)$ which is living in a Brillouin zone with $\overline{N}^2$ points where $\overline{N}$ is the GCD of N and N_ϕ. The advantage of the torus geometry is that it reveals the topological degeneracies of non-Abelian states like the Read-Rezayi parafermionic states [13, 14]. For the Bose $k = 3$ state of Eq.(2), there is a twofold degeneracy of the ground state which is exact only in the thermodynamic limit (in addition to a global twofold degeneracy of the center of mass). The corresponding doublet of states has $\mathbf{K} = \mathbf{0}$. The standard hierarchy/composite fermion states only have the center of
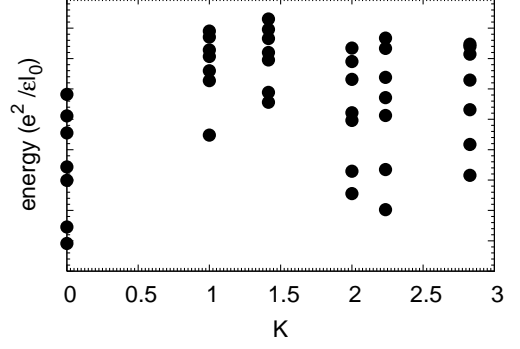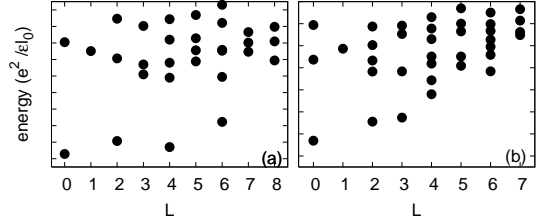
mass degeneracy which is discarded in what follows. Diagonalization of a system of $N = 12$ electrons at $N_\phi = 28$ in a square unit cell shows evidence for a doublet ground state at $\mathbf{K} = \mathbf{0}$ in the neighborhood of $\delta V_1 = -0.125$: see Fig.(3). The formation of the low-lying doublet appear before weakening of the states with nonzero wavevector, i.e. prior to an instability that breaks translation symmetry. Further weakening of V_1 leads to a compressible stripe phase beyond $\delta V_1 = -0.15$, where the spectrum changes abruptly when going to a rectangular unit cell (not shown) with an aspect ratio of 0.95.

Another state that can be studied with present exact diagonalization techniques is $\nu = 3/11$. This fraction is obseved experimentally and may be tentatively described by three filled pseudo-Landau levels of ^{4}CFs. For $N = 6$ the non-Abelian state with positive flux at $k = 3$ is realized on the sphere at $N_\phi = 17$ instead of $N_\phi = 21$ for the CF state. At this flux with pure Coulomb interaction, one finds a band of low-lying states with $\Delta L = 2$ spacing, i.e. two quasiholes states on top of the closeby 1/3 liquid at $N_\phi = 15$: see Fig.(4a). If one weakens the
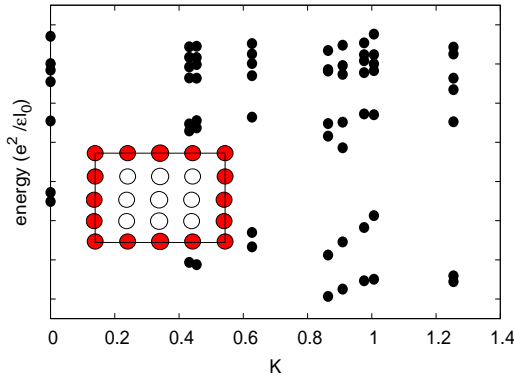
Figure 5: Low-lying spectrum vs. pseudomomentum $K = |\mathbf{K}|$ for N=12 electrons at $N_\phi = 32$ on a torus with aspect ratio 0.95 in the LLL : the sevenfold degenerate phase at $\delta V_1 = +0.15$. Inset : the Brillouin zone with the wavevectors (filled dots) of the degenerate ground states (color online).

hard-core component of the potential, then the ground state emerges clearly from the remainder of the spectrum with a possible branch of neutral excitations : see Fig.(4b).

Finally I turn to the state at $\nu = 3/8$ for which several competing states have been proposed [21, 22, 23]. To observe $k = 6$ clustering I use the torus geometry with $N = 12$ electrons and $N_\phi = 32$. The phase diagram is similar for both the LLL and the N=1 LL. There is a extended phase including the pure Coulomb point which is likely to be a kind of Fermi sea of composite particles. There is a nondegenerate ground state ruling out broken symmetry phases and the spectrum is very sensitive to change of the aspect ratio in the rectangular geometry. Reducing V_1 leads to a transition towards a bubble phase at $\delta V_1 \approx -0.18$ in the LLL ($\delta V_1 \approx -0.11$ in the N=1 LL). The degenerate ground states form a two-dimensional array in the magnetic Brillouin zone [24] and their number matches that of a bubble phase with 3 electrons per bubble : see Fig.(5). This is similar to what is observed in higher Landau levels. For larger values of V_1, the system enters a phase with a seven-fold degenerate ground state for $\delta V_1 \approx +0.10$ in the LLL and $\delta V_1 \approx +0.15$ in the N=1 LL : see Fig.(5). The wavevectors of the ground states lie at the boundary of the Brillouin zone and do not define one or two dimensional ordering : see inset of Fig.(5) so a possible interpretation is a topological degeneracy. This is what can be expected from the non-Abelian $k = 6$ state of Eq.(4) (although the wavevectors of the ground states are not known). It is yet not possible to assess the incompressibility of this state however it should be noted that it is robust to changes of the aspect ratio of the torus from unity to at least 0.85 contrary to the phase in the immediate neighborhood of the Coulomb point. Also this phase with large degeneracy survives in the case of

the pure hard-core model with V_1 only. This is very different from the physics at $\nu = 1/2$ in the LLL or the N=1 LL where the Pfaffian state is confined to a narrow range of interactions. The $k = 6$ is a new candidate for the state observed at $3/8$ in the LLL as well as the $2+3/8$ state. More detailed studies should await progress in understanding the construction of such states which at the present time does not allow for overlap calculations due to the (practical) complexity of Eq.(4).

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